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Optimal Insulation of Structures with Varying Thermal Conductivity

Ahmet Z. Sahin*
King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

The optimal distribution of a limited amount of insulation material on a composite structure is obtained. Three cases for the variation of the composite wall thermal conductivity were considered. These are 1) power, 2) linear, and 3) exponential types of variation of wall thermal conductivity with an in-plane direction along the surface of the wall. The surfaces of the wall are assumed to either have constant temperatures or be exposed to ambient at given temperatures. The variation of the optimum insulation thickness is shown to be relatively smooth and easily applicable for all three profiles of thermal conductivity. For a certain optimum length of composite structure, optimal distribution of the insulation material provides maximum percent energy savings. Optimal distribution of the insulation material may require partially covering the surface.

Nomenclature

a = $k_1 L$
b = $k t_w / k_0 \bar{\boldsymbol{\delta}}$
F = integrand of Φ
k = thermal conductivity, W/mK
k_i = thermal conductivity of insulation material, W/mK
k_0 = thermal conductivity of composite wall at $x = 0$, W/mK
k_1 = dimensional constant, 1/m
L = length of the composite wall, m
q' = total heat transfer per unit width of the composite wall, W/m
q'' = local heat flux, W/m^2
$\bar{q}''$ = average heat flux, W/m^2
$\bar{q}''_{\min}$ = average minimum heat flux, W/m^2
T = temperature, K
T_i = inner surface temperature of the composite wall, K
T_o = outer surface temperature of the composite wall, K
t_w = thickness of the composite wall, m
x = spatial coordinate, m
$\boldsymbol{\delta}$ = insulation thickness, m
$\bar{\delta}$ = average insulation thickness, m
δ_{opt} = optimum insulation thickness, m
$\boldsymbol{\lambda}$ = Lagrange multiplier
ξ = x/L
Φ = aggregate integral

I. Introduction

CONSERVING energy is one of the major concerns in many industrial applications. In this regard, the supply of insulation material, in general, has limitations because of the cost, weight, and volume of the insulation material. The cost of purchase, installation, and maintenance can often be expensive. The weight and volume of insulation material is required to be minimal in some cases such as aerospace applications. Thus, it is required to distribute the limited amount of insulation material such that an optimal insulation is achieved. This is especially important where the structure is not isotropic, such as aerospace and automotive structures, in which the conductivity could be varying throughout the wall to be insulated.

Insulation of surfaces are always made by using uniform thicknesses of the insulation layer.[1,2] Very few studies can be found in the literature regarding the optimal distribution of insulation thickness. In an effort to minimize the total heat transfer through a wall that is subjected to convective heat loss on one side, Lim et al.[3] studied the optimum wall thickness variation using variational calculus. They found that the total heat transfer rate in the case of forced convection can be minimized when the wall thickness decreases in an optimal manner in the direction of flow. Bejan[4] studied the optimal distribution of a finite amount of insulation material on a nonisothermal wall to minimize the overall heat transfer.

In this paper, the optimal distribution of a limited amount of insulation material on a composite wall with variable thermal conductivity with an in-plane direction is studied. Exact analytical solutions for the insulation thickness variation were obtained for certain classes of conductivity profiles that include power, linear, and exponential types of variations. Distributing the insulation material in an optimal manner, it is shown that the energy savings could be maximum for certain lengths of composite wall.

II. Formulation of the Problem

Consider the uniform thickness composite wall with varying thermal conductivity $k(x)$ as shown in Fig. 1. T_i and T_o are fixed. It is desired to distribute the finite amount of insulation material with k_i so that the heat transfer through the wall becomes minimum. That is, the optimum insulation thickness δ_{opt} is required.

Because of the possible large variation of the thermal conductivity of the composite material, it may be necessary that the insulation material only partially cover the surface to minimize the heat transfer. Then, part of the surface that extends from the leading edge up to the location $x = x_c$ may be left uninsulated. The thickness of the composite wall is assumed

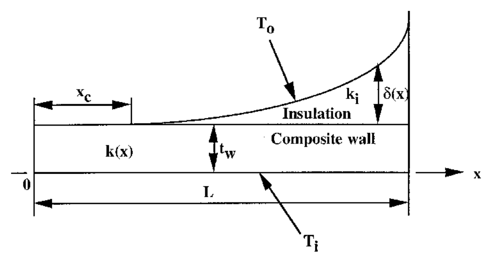

Fig. 1 Composite wall with varying insulation thickness.

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*Associate Professor, Department of Mechanical Engineering.

to be small enough to neglect the heat transfer with an in-plane direction. Therefore, the local one-dimensional heat flux through the wall and the layer of insulation is given by

$$q'' = \frac{\Delta T}{[t_w/k(x)] + [\delta(x)/k_i]} \tag{1}$$

where ΔT is the temperature difference of inner and outer surfaces of the wall (or wall plus insulation) and the thickness of the insulation $\delta(x)$ can take values of either zero or positive. It should be noted that the contact resistance between the composite wall and the insulation is neglected in writing Eq. (1). However, possible contact resistance can easily be accounted for by adding a constant thermal contact resistance term to the bulk thermal resistance of the insulation. Integrating Eq. (1) along the entire length L, q' is obtained as

$$q' = \int_0^{x_c} k(x)\frac{\Delta T}{t_w}\,\mathrm{d}x + \int_{x_c}^{L}\frac{\Delta T}{[t_w/k(x)] + [\delta(x)/k_i]}\,\mathrm{d}x \tag{2}$$

where $x_c \geq 0$.

The total volume of insulation material available per unit width of the wall is

$$\int_{x_c}^{L}\delta(x)\,\mathrm{d}x = \bar{\delta}L = \text{const} \tag{3}$$

The minimization of q' in Eq. (2) subject to the volume constraint given by Eq. (3) can be achieved by minimization of the aggregate integral,[5] given by

$$\Phi = \int_{x_c}^{L}\left\{\frac{\Delta T}{[t_w/k(x)] + [\delta(x)/k_i]} + \lambda\delta(x)\right\}\mathrm{d}x = \int_{x_c}^{L} F\,\mathrm{d}x \tag{4}$$

The optimum thickness is obtained by solving the Euler equation that is

$$\frac{\partial F}{\partial \delta} = 0 \tag{5}$$

and the solution for the optimum thickness of the insulation can be found as

$$\delta_{\mathrm{opt}}(x) = \sqrt{(k_i\Delta T/\lambda)} - k_i[t_w/k(x)] \tag{6}$$

λ can be evaluated by using the volume constraint, Eq. (3)

$$\sqrt{\frac{k_i\Delta T}{\lambda}} = \frac{L}{L - x_c}\left(\bar{\delta} + \frac{1}{L}\int_0^{L} k_i\frac{t_w}{k(x)}\,\mathrm{d}x\right) \tag{7}$$

Thus, the optimum thickness of insulation becomes

$$\delta_{\mathrm{opt}}(x) = \frac{L}{L - x_c}\left[\bar{\delta} + \frac{1}{L}\int_0^{L} k_i\frac{t_w}{k(x)}\,\mathrm{d}x - k_i\frac{t_w}{k(x)}\right] \tag{8}$$

which is a function of the thermal conductivity variation of the wall $k(x)$. It should be noted that if the wall thermal conductivity is uniform, Eq. (8) reduces to $\delta_{\mathrm{opt}}(x) = \bar{\delta} = \text{const}$ as expected.

Exact analytical solutions of optimum insulation thickness for the power-, linear-, and exponential-types of thermal conductivity variation of the composite wall are obtained in the following. The thermal conductivity profiles have the following form.

Power type:

$$k(x) = k_0(1 + k_1x)^{(m-1)/m}$$

Linear type:

$$k(x) = k_0(1 + k_1x)$$

Exponential type:

$$k(x) = k_0\exp(k_1x)$$

in which m is a real constant, and k_1 is a real dimensional parameter.

III. Power-Type Thermal Conductivity

Substituting the power-type thermal conductivity profile in Eq. (8)

$$\delta_{\mathrm{opt}}(x) = \frac{L}{L - x_c}\left\{\bar{\delta} + m\frac{k_it_w}{k_0k_1L}[(1 + k_1L)^{1/m} - (1 + k_1x_c)^{1/m}]\right\} - \frac{k_it_w}{k_0}(1 + k_1Lx)^{(1-m)/m} \tag{9}$$

Equation (9) can be written in nondimensionalized form as

$$\frac{\delta_{\mathrm{opt}}(\xi)}{\bar{\delta}} = \frac{1}{1 - \xi_c}\left\{1 + m\frac{b}{a}[(1 + a)^{1/m} - (1 + a\xi_c)^{1/m}]\right\} - b(1 + a\xi)^{(1-m)/m} \tag{10}$$

where $\xi = x/L$, $\xi_c = x_c/L$, $a = k_1L$, and $b = k_it_w/k_0\bar{\delta}$.

The minimum average one-dimensional heat flux that corresponds to this optimum insulation thickness can be obtained by

$$\bar{q}''_{\min} = \frac{1}{L}\left\{\int_0^{x_c} k(x)\frac{\Delta T}{t_w}\,\mathrm{d}x + \int_{x_c}^{L}\frac{\Delta T}{[t_w/k(x)] + [\delta_{\mathrm{opt}}(x)/k_i]}\,\mathrm{d}x\right\} \tag{11}$$

Substituting Eq. (9) into Eq. (11) and carrying out the integration and using the power type of thermal conductivity variation, the minimum average heat flux obtained as a function of the previously dimensionless parameters is

$$\frac{\bar{q}''_{\min}}{(k_i\Delta T/\bar{\delta})} = \frac{m}{ab(2m-1)}[(1 + a\xi_c)^{(2m-1)/m} - 1] + \frac{a(1-\xi_c)^2}{a + mb[(1 + a)^{1/m} - (1 + a\xi_c)^{1/m}]} \tag{12}$$

On the other hand, the average heat flux in the case of uniform insulation thickness $\delta(x) = \bar{\delta}$ is

$$\frac{\bar{q}''}{(k_i\Delta T/\bar{\delta})} = \int_0^1\frac{\mathrm{d}\xi}{1 + b(1 + a\xi)^{(1-m)/m}} \tag{13}$$

The solution of Eq. (13) for any arbitrary value of m is possible numerically. However, exact solutions are possible for some special cases as given in the following sections.

A. Case: $m = 2$ (Square-Root Profile)

This case corresponds to the square-root thermal conductivity profile

$$k(x) = k_0\sqrt{1 + k_1 x} \tag{14}$$

for which the optimum insulation thickness is obtained as

$$\delta_{\text{opt}}(\xi)/\bar{\delta} = \frac{1}{1 - \xi_c}[1 + 2(b/a)(\sqrt{1+a} - \sqrt{1 + a\xi_c})] - [b/(\sqrt{1 + a\xi})] \tag{15}$$

The minimum average one-dimensional heat flux in this case is

$$\frac{\bar{q}''_{\min}}{(k_i\Delta T/\bar{\delta})} = \frac{2}{3ab}[(1 + a\xi_c)^{3/2} - 1] + \frac{a(1-\xi_c)^2}{a + 2b(\sqrt{1+a} - \sqrt{1 + a\xi_c})} \tag{16}$$

and the average heat flux for uniform insulation thickness is obtained by integrating Eq. (13) for $m = 2$

$$\frac{\bar{q}''}{(k_i\Delta T/\bar{\delta})} = 1 + 2\frac{b}{a}(1 - \sqrt{1+a}) + \frac{b^2}{a}\ell n\left[\frac{1 + a + b^2 + 2b\sqrt{1+a}}{(1+b)^2}\right] \tag{17}$$

B. Case: $m = -1$ (Quadratic Profile)

This case corresponds to the quadratic thermal conductivity profile

$$k(x) = k_0(1 + k_1 x)^2 \tag{18}$$

for which the optimum insulation thickness is obtained as

$$\frac{\delta_{\text{opt}}(\xi)}{\bar{\delta}} = \frac{1}{1 - \xi_c}\left[1 - \frac{b}{a}\left(\frac{1}{1+a} - \frac{1}{1 + a\xi_c}\right)\right] - \frac{b}{(1 + a\xi)^2} \tag{19}$$

The minimum average one-dimensional heat flux in this case is

$$\frac{\bar{q}''_{\min}}{(k_i\Delta T/\bar{\delta})} = \frac{1}{3ab}[(1 + a\xi_c)^3 - 1] + \frac{a(1-\xi_c)^2}{a - b[[1/(1+a)] - [1/(1 + a\xi_c)]]} \tag{20}$$

and the average heat flux for uniform insulation thickness is obtained by integrating Eq. (13) for $m = -1$

$$\frac{\bar{q}''}{(k_i\Delta T/\bar{\delta})} = 1 + \frac{\sqrt{b}}{a}\left[\arctan\left(\frac{1}{\sqrt{b}}\right) - \arctan\left(\frac{1+a}{\sqrt{b}}\right)\right] \tag{21}$$

IV. Linear-Type Thermal Conductivity

Substituting the linear-type thermal conductivity profile in Eq. (8) and carrying out the integration, the optimal insulation thickness in this case is obtained as

$$\frac{\delta_{\text{opt}}(\xi)}{\bar{\delta}} = \frac{1}{1 - \xi_c}\left[1 + \frac{b}{a}\ell n\left(\frac{1+a}{1 + a\xi_c}\right)\right] - \frac{b}{(1 + a\xi)} \tag{22}$$

The minimum average one-dimensional heat flux that corresponds to this optimum insulation thickness can be obtained by substituting Eq. (22) in Eq. (11) and performing the integration with linear-type thermal conductivity profile as

$$\frac{\bar{q}''_{\min}}{(k_i\Delta T/\bar{\delta})} = \frac{\xi_c(2 + a\xi_c)}{2b} + \frac{a(1-\xi_c)^2}{a + b\,\ell n[(1+a)/(1 + a\xi_c)]} \tag{23}$$

and $\delta(x) = \bar{\delta}$ is

$$\frac{\bar{q}''}{(k_i\Delta T/\bar{\delta})} = 1 + \frac{b}{a}\ell n\left(\frac{1+b}{1+a+b}\right) \tag{24}$$

It should be noted that the case of linear-type thermal conductivity is in fact another special case of power-type thermal conductivity when $m \to \infty$.

V. Exponential-Type Thermal Conductivity

Similarly, substituting the exponential-type thermal conductivity profile in Eq. (8) and carrying out the integration, the optimal insulation thickness in this case is obtained as

$$\delta_{\text{opt}}(\xi)/\bar{\delta} = [1/(1 - \xi_c)][1 + (b/a)[\exp(-a\xi_c) - \exp(-a)]] - b\exp(-a\xi) \tag{25}$$

The minimum average one-dimensional heat flux that corresponds to this optimum insulation thickness can be obtained by substituting Eq. (25) in Eq. (11) and performing the integration with a exponential-type thermal conductivity profile as

$$\frac{\bar{q}''_{\min}}{(k_i\Delta T/\bar{\delta})} = \frac{\exp(a\xi_c) - 1}{ab} + \frac{a(1-\xi_c)^2}{a + b[\exp(-a\xi_c) - \exp(-a)]} \tag{26}$$

and the $\delta(x) = \bar{\delta}$ is

$$\frac{\bar{q}''}{(k_i\Delta T/\bar{\delta})} = \frac{1}{a}\ell n\left[\frac{b + \exp(a)}{1+b}\right] \tag{27}$$

VI. Energy Savings

The percent savings of energy by distributing the insulation in the optimal way can be calculated for each of the cases given previously by

$$\%\ \text{savings} = [1 - (\bar{q}''_{\min}/\bar{q}'')] \times 100 \tag{28}$$

VII. Discussion

The thermal conductivity profiles of the composite wall considered in this study are given in Fig. 2. The exponential- and the quadratic-type thermal conductivity profiles give much higher thermal conductivity values toward the end of the com-

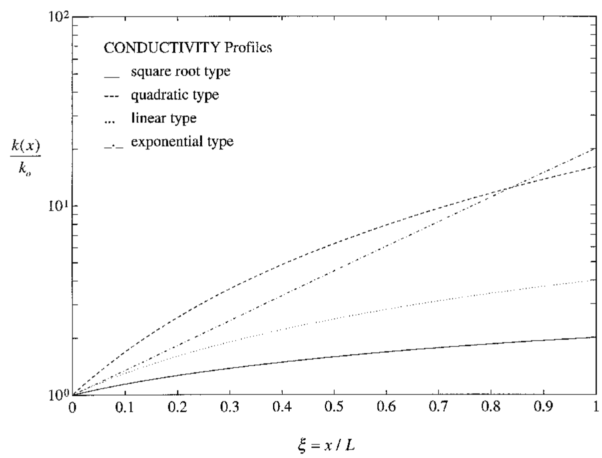

Fig. 2 Thermal conductivity profiles of the composite wall for a = 3 and b = 1.

posite wall than those obtained by the other profiles. Therefore, it may be expected that the insulation thicknesses $\delta_{opt}(\xi)/\bar{\delta}$ should also be much higher in the case of exponential and quadratic profiles toward the end of the composite wall. However, as can be seen from Fig. 3, this is not the case. The insulation thicknesses for the exponential- and quadratic-type thermal conductivity profiles at the same location are not much different from the others.

As far as the leading edge of the composite wall is concerned, the insulation thicknesses vary considerably from one another, being consistent with the thermal conductivity profiles considered.

Figure 3 shows that the insulation thicknesses $\delta_{opt}(\xi)/\bar{\delta}$ in each case are quite smooth. Therefore, insulating the composite wall in the optimal way can easily be made no matter how sharp variation in thermal conductivity exists. The thickness of the insulation is obtained to be positive everywhere along the surface of the composite for $a = 3$ and $b = 1$.

For given parameters of thermal conductivity of the composite wall and insulation material, distribution of the insulation material may provide better savings for certain sizes of the composite walls. Figure 4 shows the insulation thickness, at the end of the composite wall, $\delta_{opt}(\xi = 1)/\bar{\delta}$, for different lengths of the wall, $a = k_1L$. It is clear that the thickness of the insulation material at end of the composite wall, $\delta_{opt}(\xi = 1)/\bar{\delta}$, is maximum at certain lengths of the composite wall, depending on the thermal conductivity profile and selected parameters. This means that varying the insulation thickness in the optimal way becomes more important if the length of the composite wall happens to be around the critical value, which lies somewhere between $a = k_1L = 1$ to 5 in Fig. 4.

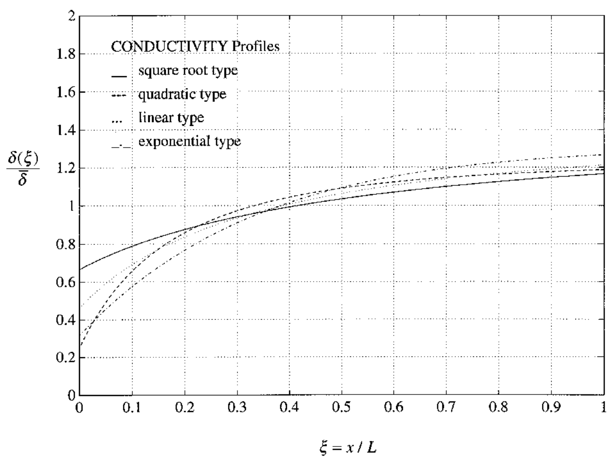

Fig. 3 Optimum insulation thicknesses for $a = 3$ and $b = 1$.

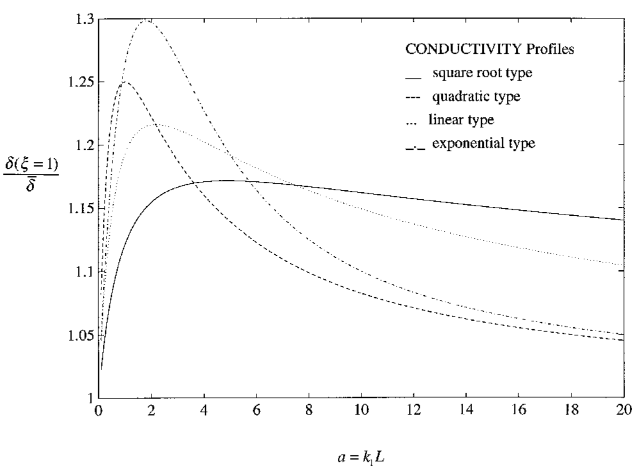

Fig. 4 Variation of insulation thickness at $\xi = x/L = 1$ as function of $a = k_1L$ for different thermal conductivity profiles and $b = 1$.

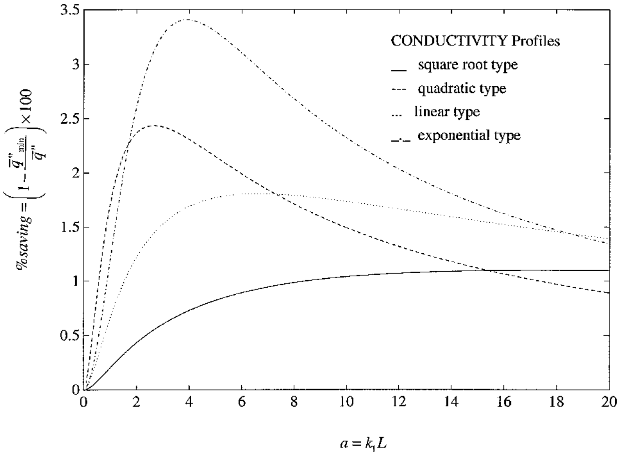

Fig. 5 Energy saving by optimal distribution of insulation material on the composite wall as function of $a = k_1L$ for different thermal conductivity profiles and $b = 1$.

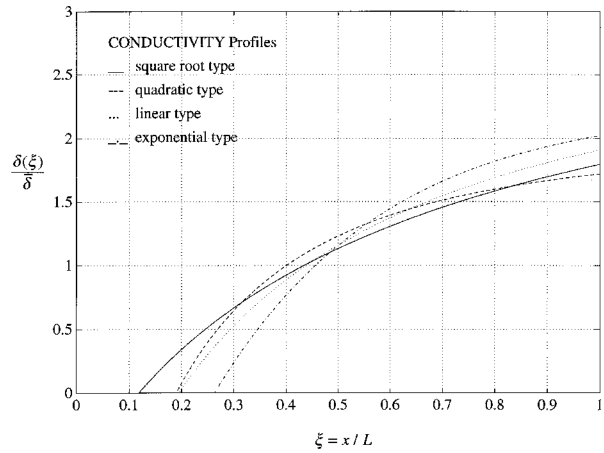

Fig. 6 Optimum insulation thicknesses for $a = 3$ and $b = 5$.

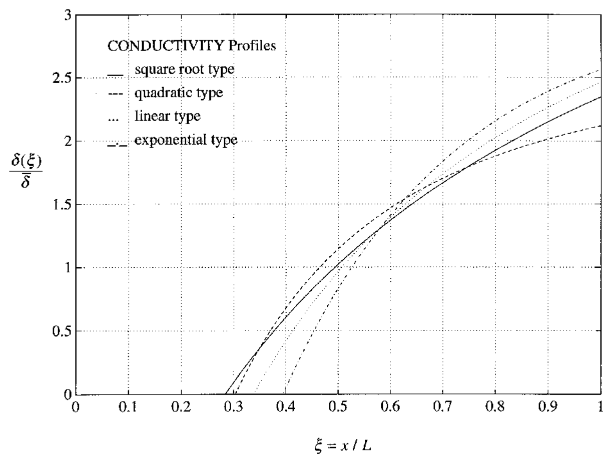

Fig. 7 Optimum insulation thicknesses for $a = 3$ and $b = 10$.

The percent energy savings for each of the profiles considered are given in Fig. 5 as a function of the length of the composite wall, $a = k_1L$, for $b = 1$. Again, it is obvious that there exists a critical length at which maximum energy savings is possible when insulation is made in the optimal way. This is especially important when the thermal conductivity of the composite wall $k(x)$ varies in an exponential or quadratic manner.

Table 1 Uninsulated fraction of composite surface ξ_c in optimal distribution of insulation material (a = 1)

	Type of thermal conductivity profile				
b	Cubic, $m = -0.5$	Square root, $m = 2$	Quadratic, $m = -1$	Linear	Exponential
1	0	0	0	0	0
2	0.0519	0	0.0532	0.0132	0.0701
3	0.0959	0	0.1111	0.0905	0.1579
4	0.1291	0.0657	0.1547	0.1483	0.2179
5	0.1559	0.1185	0.1897	0.1943	0.2630
6	0.1785	0.1622	0.2190	0.2322	0.2988
7	0.1980	0.1992	0.2440	0.2643	0.3284
8	0.2152	0.2312	0.2660	0.2920	0.3536
9	0.2306	0.2593	0.2855	0.3163	0.3753
10	0.2445	0.2841	0.3030	0.3379	0.3944

Table 2 Percent savings in optimal distribution of insulation material (a = 3)

	Type of thermal conductivity profile				
b	Cubic, $m = -0.5$	Square root, $m = 2$	Quadratic, $m = -1$	Linear	Exponential
0.5	0.7628	0.2400	0.8125	0.5610	1.1177
1	2.3653	0.6036	2.4224	1.5292	3.2725
2	6.0315	1.2096	5.9491	3.3837	7.9112
3	8.8427	1.6313	8.5255	4.6966	11.3189
4	10.9246	1.9149	10.2633	5.4464	13.5939
5	12.5291	2.0765	11.4698	5.8525	15.1678
6	13.8074	2.1560	12.3297	6.0577	16.2940
7	14.8515	2.1851	12.9533	6.1426	17.1194
8	15.7207	2.1834	13.4095	6.1535	17.7336
9	16.4547	2.1629	13.7436	6.1182	18.1946
10	17.0819	2.1311	13.9864	6.0538	18.5410

The parameter $b = k_i t_w / k_0 \bar{\delta}$ can be interpreted as the ratio of the wall thermal resistance at the origin ($\xi = 0$) and the average thermal resistance of the insulation. Since the thermal conductivity of the composite wall is minimum at the origin, $k(x = 0) = k_0$, and increasing along ξ in all of the thermal conductivity profiles that are used, the actual ratio of the wall thermal resistance and the average thermal resistance of the insulation decreases with ξ. Therefore, in the case $b = 1$, the ratio of wall to average insulation thermal resistances is equal to 1 at $\xi = 0$ and decreases along ξ, reaching a value as low as 0.05. On the other hand, for t_w and k_0, large b values correspond to the case of availability of only a small amount of insulation material (small $\bar{\delta}$) to be optimally distributed on the surface of the wall. To see the effect of the parameter b, the insulation thickness along the surface of the composite material is given in Figs. 6 and 7 for $b = 5$ and $b = 10$, respectively. Optimization of insulation suggests that part of the composite material should be left uninsulated and all of the available insulation material should be distributed in the region $\xi_c < \xi < 1$. It should be noted that as parameter b decreases, the optimum thickness of the insulation becomes more uniform, as can be seen by comparing Figs. 3, 6, and 7. This means that the optimal distribution of the available insulation material becomes more important for large values of parameter b (i.e., when only a small amount of insulation material is available).

The location of the start of insulation $\xi = \xi_c$ and the percent savings of energy by optimal insulation is given in Tables 1 and 2, respectively, for different values of b. It is clear that considerable amount of energy savings is possible when the insulation material is distributed in the optimal way.

VIII. Conclusions

Distribution of a limited amount of insulation material on a composite surface with varying thermal conductivity is studied analytically to minimize the total heat loss from the surface. Three types of thermal conductivity profiles, 1) power, 2) linear, and 3) exponential, were used in the analysis. It is found that the optimum insulation thickness variation is smooth and easily applicable even for the composite walls with considerably varying thermal conductivity. The optimum thickness of insulation may show significant variation, especially for exponential- and quadratic- (power-) types of thermal conductivity profiles of composite walls. There exists an optimum length of the composite wall for which optimum distribution of the insulation material provides the maximum percent energy savings. Optimization of insulation may suggest that the limited amount of insulation material be distributed in a way to partially cover the surface.

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